drawn from O, form a triangle OPR with the angle φ at O. Turn this triangle about O through an angle $= \measuredangle POQ$ so that it takes the position $OP'R'$; a parallel through Q to $P'R'$ then determines $\ddot{q} = Q' - O$. Adding this to $\ddot{O}$ we find $\ddot{Q}$.

16. Finally, by using projections somewhat as in Art. 7 for velocities, we can derive $\ddot{Q}$ from $\ddot{O}$ and $\ddot{P}$ as follows.

Let OO_1 and OP_1 be the projections on OP of $\ddot{O}$ and $\ddot{P}$, drawn from O (Fig. 9), and construct on P_1O_1 the triangle $P_1O_1Q_1$ similar to OPQ. Let the parallel to OP through Q_1 meet OQ at Q', PQ at Q''; and let OO' be the projection of $\ddot{O}$ on OQ, PP' the projection of $\ddot{P}$ on PQ.

Then $Q'O'$ is the projection of $\ddot{O}$ on QO, $Q''P'$ the projection of $\ddot{Q}$ on QP; transferring these to Q, $\ddot{Q}$ is found.

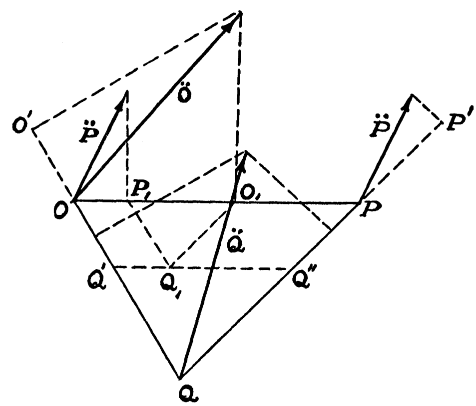

FIG. 9.

The constructions of Arts. 7, 11–13, 16 were first obtained from the equations in Cartesian coordinates, OP being taken as axis of x. Using the vector method it is naturally preferable to deal with the vectors themselves rather than with their projections.

NOTE ON CERTAIN ALGEBRAIC EQUATIONS.

By HERMON L. SLOBIN, University of Minnesota.

In elementary analysis we often meet problems like the following: Show that the roots of the equation

I. $$x^3 + x^2 - 2x - 1 = 0 \quad \text{are} \quad 2\cos\frac{2}{7}\pi, \quad 2\cos\frac{4}{7}\pi, \quad 2\cos\frac{8}{7}\pi.$$

Show that the roots of the equation

II. $$x^3 - 5x^2 + 6x - 1 = 0 \quad \text{are} \quad 4\cos^2\frac{\pi}{7}, \quad 4\cos^2\frac{2\pi}{7}, \quad 4\cos^2\frac{4\pi}{7}.$$

These problems are taken from Bromwich, "Theory of Infinite Series," page 188. The method of solution used by Bromwich involves a knowledge of certain formulæ in the theory of trigonometric series. These formulæ are based upon the consideration of a certain differential equation, and the method of solution by their means is applicable only when these roots of the algebraic equations have very special forms.

Problems may be stated which require algebraic equations to be set up whose roots are given trigonometric forms such as the above. In a previous paper[1] it has been shown that the trigonometric functions whose arguments are rational multiples of π, are algebraic numbers. The general problem would then be to find the algebraic equation whose roots are $\theta_1(r_1\pi)$, $\theta_2(r_2\pi)$, $\cdots$ $\theta_n(r_n\pi)$, where θ_i represents any trigonometric function, $i = 1, 2, \cdots n$, and r_i represents any rational number. More generally, θ_i may represent any expression built up of the trigonometric functions whose arguments are rational multiples of π, by means of the operations of addition, subtraction, multiplication, division, raising to powers, and extracting roots.

A formal elementary method is readily obtainable for the solution of such problems, but in the general case the operations of reducing the resultant equation to an algebraic equation with rational coefficients would be quite cumbersome, and the degree of the final equation often much larger than the number of roots assigned as such trigonometric forms. However, in particular cases, where the degree of the rational algebraic equation is the same as the number of roots assigned as trigonometric forms, the operations are simple, and the *process* is applicable even in the general case.

I shall illustrate this method by applying it to the problems stated above. The detailed operations are instructive and the forms in which the coefficients enter in the equation are of especial interest.

1. Given the roots $a_1 = 2\cos\frac{2\pi}{7}$, $a_2 = 2\cos\frac{4\pi}{7}$, $a_3 = 2\cos\frac{8\pi}{7}$.

From De Moivre's theorem, $(\cos x + i \sin x)^r = \cos rx + i \sin rx$, when $x = \pi$, and r is any rational number, we have

$$\cos r\pi = \frac{(-1)^{2r} + 1}{2(-1)^r},$$

where $(-1)^r$ denotes *any of the complex roots*. That is, if $r = p/q$, $(-1)^{p/q}$ means the pth power of any of the complex qth roots of -1. Hence

$$a_1 = \frac{(-1)^{\frac{4}{7}} + 1}{(-1)^{\frac{2}{7}}}, \quad a_2 = \frac{(-1)^{\frac{8}{7}} + 1}{(-1)^{\frac{4}{7}}}, \quad a_3 = \frac{(-1)^{\frac{16}{7}} + 1}{(-1)^{\frac{8}{7}}}.$$

The equation

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 = 0$$

will now be constructed.

Bauer and Slobin, *Rendiconti*, 1913.

Multiplying numerator and denominator of a_1 by $(-1)^{\frac{5}{7}}$ and reducing the numerator by substituting (-1) for $(-1)^{\frac{7}{7}}$, we have

$$a_1 = (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}};$$

similarly

$$a_2 = (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}};$$
$$a_3 = -(-1)^{\frac{1}{7}} + (-1)^{\frac{6}{7}}.$$

Hence

$$a_1 + a_2 + a_3 = [(-1)^{\frac{6}{7}} - (-1)^{\frac{5}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + (-1)^{\frac{2}{7}} - (-1)^{\frac{1}{7}}].$$

We will denote this bracket by K. From $x^7 + 1 = 0$ we have for all the complex roots of $(-1)^{\frac{1}{7}}$,

$$x^6 - x^5 + x^4 - x^3 + x^2 - x = -1.$$

Hence

$$K = -1.$$

Likewise

$$a_1a_2 = (-1)^{\frac{6}{7}} + (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}} - (-1)^{\frac{1}{7}},$$
$$a_1a_3 = -(-1)^{\frac{3}{7}} + (-1)^{\frac{6}{7}} - (-1)^{\frac{1}{7}} + (-1)^{\frac{4}{7}},$$
$$a_2a_3 = -(-1)^{\frac{5}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + (-1)^{\frac{2}{7}}.$$

Hence

$$a_1a_2 + a_1a_3 + a_2a_3 = 2K = -2,$$

and finally

$$a_1a_2a_3 = 2 + K = 1.$$

Therefore the required equation is

$$x^3 + x^2 - 2x - 1 = 0.$$

2. Given the roots $a_1 = 4\cos^2\dfrac{\pi}{7}$, $a_2 = 4\cos^2\dfrac{2\pi}{7}$, $a_3 = 4\cos^2\dfrac{4\pi}{7}$. By a process similar to the above, we have

$$a_1 = (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}} + 2,$$
$$a_2 = (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + 2.$$
$$a_3 = -(-1)^{\frac{1}{7}} + (-1)^{\frac{6}{7}} + 2.$$

Also,

$$a_1 + a_2 + a_3 = 6 + K = 5,$$
$$a_1a_2 + a_1a_3 + a_2a_3 = 12 + 6K = 6,$$
$$a_1a_2a_3 = 10 + 9K = 1.$$

Hence the equation is

$$x^3 - 5x^2 + 6x - 1 = 0.$$

As a further example a case involving somewhat more difficult rationalizations is proposed by the author in the problem department in this issue, namely, to find the algebraic equation whose roots are

$$a_1 = \cos\frac{\pi}{9},\ a_2 = -\cos\frac{2\pi}{9},\ a_3 = -\cos\frac{4\pi}{9}.$$